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LETTER TO THE EDITOR

Intrinsic fluctuations determined by the existence of a centre manifold

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Abstract. An experiment measuring the transition to a convective pattern in a Rayleigh-Bénard cell is shown to furnish evidence supporting the fact that the onset of a centre manifold determines scaling equations relating the intensity of intrinsic fluctuations to the other characteristic small parameters of the system.

The identification of synergistic organisations with a locally attractive locally invariant centre manifold (CM) emerging beyond a dynamic instability has been shown to explain successfully the contraction in phase space occurring near a dynamic critical point [1, 2]. The onset of a CM in phase space corresponds to the statistical subordination of fast-relaxing modes to order parameters as is verified in dissipative systems. The order parameters are the CM coordinates. In order for this structure to be sustained, a balance exists between the fast deterministic drift towards the CM and the diffusive pressure provided by the intrinsic fluctuations. This competition determines the probability distribution about the CM [2-4]. Thus the strength of the statistical bath is not given by the intensity of equilibrium thermal fluctuations [5] but it is related to the unfolding parameters of the system [4].

To pose the problem adequately, let $\mathbf{T}$ be a transformation associated to the CM reduction of fast modes. That is, the system transformed under $\mathbf{T}$ is in Poincaré-Jordan normal form [1, 2]. If $\boldsymbol{X}$ is the stochastic vector field describing the system near the threshold, then the following decomposition is unique:

$$V = \mathbf{T}\boldsymbol{X} = \boldsymbol{X}_s + \boldsymbol{X}_f + \mathbf{T}\boldsymbol{f} \tag{1}$$

where $\boldsymbol{X}_s$ is the vector of order parameters, $\boldsymbol{X}_f$, the vector of fast-relaxing modes and $\boldsymbol{f}$, the random source term. A stochastic order parameter equation is obtained by integrating the general Fokker-Planck equation for the probability functional $P(\boldsymbol{X}_s, \boldsymbol{X}_f, t)$ along the CM:

$$\int_{\mathrm{CM}} \partial_t P \, \mathrm{d}\boldsymbol{X}_f = \int_{\mathrm{CM}} \bigg(-\sum_i \partial_{X_{s,i}} \{ (\dot{\boldsymbol{X}}_{s,i} - (\mathbf{T}\boldsymbol{f})_{s,i}) P \} - \sum_j \partial_{X_{f,j}} \{ (\dot{\boldsymbol{X}}_{f,j} - (\mathbf{T}\boldsymbol{f})_{f,j}) P \} + \tfrac{1}{2} \sum_{i,i'} C_{ii'} \, \partial^2_{X_{s,i} X_{s,i'}} P + \sum_{i,j} C_{ij} \, \partial^2_{X_{s,i} X_{f,j}} P + \sum_{j,j'} \tfrac{1}{2} C_{jj'} \, \partial^2_{X_{f,j} X_{f,j'}} P \bigg) \mathrm{d}\boldsymbol{X}_f \tag{2}$$

where the C_{ij} are the bare coupling diffusion coefficients. The strength of the statistical bath given by the random source term $\boldsymbol{f}$ should be fixed in order to obtain the proper CM-reduced equation whose drift part represents the onset of the dissipative structure.

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In order to illustrate these ideas, we shall concentrate on a specific example: the CM-reduced equation describing the transition to a convective pattern in a Rayleigh-Bénard cell swept through its threshold by means of a controlled heat input from the bottom plate [5-7].

We next concentrate on the case of a step in the heat input of the bottom plate of the Rayleigh-Bénard cell [5-7]. Following standard notation [5, 6] we write

$$\boldsymbol{X}_f = \sum_{i\geq 2} \sum_{|\boldsymbol{q}|=q_0} V_{\boldsymbol{q}}^{(i)} \boldsymbol{e}_{\boldsymbol{q}}^{(i)} \tag{3}$$

$$\boldsymbol{X}_s = \sum_{|\boldsymbol{q}|=q_0} V_{\boldsymbol{q}}^{(1)} \boldsymbol{e}_{\boldsymbol{q}}^{(1)} \tag{4}$$

where $\boldsymbol{V} = (\theta, \boldsymbol{u}, w)$, $\boldsymbol{u} = (u, v)$, $(u, v, w) =$ velocity field and $\theta =$ deviation of the temperature from the linear conducting profile between the boundaries $z = 0$, $z = 1$. The field $\boldsymbol{V}$ obeys the Boussinesq equations [5]. The distance, time and temperature are scaled respectively by d, d^2/κ, $\kappa\nu/\alpha g d^3$, where d is the cell height, κ and ν are the thermal and viscous diffusivities and α is the thermal expansion coefficient. The eigenvectors $\boldsymbol{e}_{\boldsymbol{q}}^{(i)}$ depend on the vertical coordinate z and are proportional to $\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r})$ where $\boldsymbol{r}$ is the horizontal vector, $\boldsymbol{q}$ is a horizontal wavevector and $\boldsymbol{q}_0$ is the critical wavevector for convective onset. The linear self-adjoint Boussinesq operator with eigenvectors $\boldsymbol{e}_{\boldsymbol{q}}^{(i)}$, $i \geq 1$, is $\mathbf{D}^0$ defined as [5]

$$\mathbf{D}^0 = \begin{bmatrix} \nabla^2 + \dfrac{\partial^2}{\partial z^2} & 0 & R_c \\ 0 & \sigma\left[\nabla^2 + \dfrac{\partial^2}{\partial z^2}\right] & 0 \\ \sigma & 0 & \sigma\left[\nabla^2 + \dfrac{\partial^2}{\partial z^2}\right] \end{bmatrix}. \tag{5}$$

The gradient ∇ refers to the horizontal vector $\boldsymbol{r}$, R_c is the critical Rayleigh number and σ is the Prandtl number. Free boundary conditions are assumed [5, 6]. The Fourier coordinates are defined in the canonical way:

$$V_{\boldsymbol{q}}^{(i)} = \langle \boldsymbol{e}_{\boldsymbol{q}}^{(i)}, \boldsymbol{V}\rangle. \tag{6}$$

The inner product $\langle \boldsymbol{V}_1, \boldsymbol{V}_2\rangle$ is defined by

$$\langle \boldsymbol{V}_1, \boldsymbol{V}_2\rangle = [\sigma\theta_1^*\theta_2 + R_c(\boldsymbol{u}_1^* \cdot \boldsymbol{u}_2 + w_1^* w_2)]_m. \tag{7}$$

The symbol $[\ ]_m$ indicates that we are averaging over a layer. The Nusselt number N is determined from the convective heat flow which is given by $(N-1)R/R_c$. Thus, we have

$$(N-1)R/R_c = c^2\|\boldsymbol{X}_s\|^2. \tag{8}$$

The constant c will be given later. The scaling relations will be given in terms of the small parameter ε defined as

$$\varepsilon = (R - R_c)/R_c. \tag{9}$$

In order to integrate (2), we make use of the following factorisation of P:

$$P = Q_f(\{V_{\boldsymbol{q}'}^{(i)}\}_{q',i}) Q_s(\{V_{\boldsymbol{q}}^{(1)}\}_q, t) \tag{10}$$

$$Q_f = \prod_{|\boldsymbol{q}|=q_0} \prod_{i\geq 2} (g_{\boldsymbol{q}}^{(i)}/\pi)^{1/2} \exp\{-g_{\boldsymbol{q}}^{(i)}[V_{\boldsymbol{q}}^{(i)} - \langle\!\langle V_{\boldsymbol{q}}^{(i)}\rangle\!\rangle]^2\} \tag{11}$$

where the Gaussian widths $g_{\boldsymbol{q}}^{-1/2}$, to be calculated, determine the size of fluctuations about the CM. The symbol $\langle\!\langle\ \rangle\!\rangle$ denotes the average over an ensemble of realisations of the random source field f. That is, the CM equations are given by

$$\langle\!\langle V_{\boldsymbol{q}}^{(i)} \rangle\!\rangle = F_{\boldsymbol{q}}^{(i)}(\{V_{\boldsymbol{q}}^{(1)}\}) \tag{12}$$

where the $F_{\boldsymbol{q}}^{(i)}$ are analytic functions which represent the statistical subordination of the fast variables [1-4].

Such functions can be obtained to a first approximation by the adiabatic elimination method, i.e. they are determined by means of the implicit function theorem making $\dot{V}_{\boldsymbol{q}}^{(j)} = 0, j \geqslant 2$.

The coefficients C_{ij} will be factorised in terms of the diffusion coefficients:

$$C_{ij} = d_{\boldsymbol{q}}^{(i)} d_{\boldsymbol{q}}^{(j)}. \tag{13}$$

In order to properly display the relative size of each term in the CM-reduced equation, the diffusion coefficients $d_{\boldsymbol{q}}^{(j)}$ are factorised as $d_{\boldsymbol{q}}^{(j)} = k\tilde{d}_{\boldsymbol{q}}^{(j)}$, where k is a small parameter to be properly scaled and the quantity bearing a tilde is of the order of unity.

Making use of equations (10) and (11) and the notation introduced in the preceding discussion equation (2) is

$$\begin{aligned}
\partial_t Q_s = &- \sum_{|\boldsymbol{q}|=q_0} \left[\partial_{V_{\boldsymbol{q}}^{(1)}} [\langle\!\langle \dot{V}_{\boldsymbol{q}}^{(1)} \rangle\!\rangle Q_s] + \langle\!\langle \dot{V}_{\boldsymbol{q}}^{(1)} \rangle\!\rangle \sum_{j \geqslant 2 \boldsymbol{q}'} \frac{\partial_{V_{\boldsymbol{q}}^{(1)}} g_{\boldsymbol{q}'}^{(j)}}{2 g_{\boldsymbol{q}'}^{(j)}} Q_s \right] \\
&- \sum_{|\boldsymbol{q}|=q_0} \sum_{j \geqslant 2} [\partial_{V_{\boldsymbol{q}}^{(j)}} \langle\!\langle \dot{V}_{\boldsymbol{q}}^{(j)} \rangle\!\rangle] Q_s \\
&- \sum_{|\boldsymbol{q}|=q_0} \sum_{j \geqslant 2} 2k^2 \tilde{d}_{\boldsymbol{q}}^{(j)^2} g_{\boldsymbol{q}}^{(j)} Q_s + \sum_{\boldsymbol{q},\boldsymbol{q}'} \sum_{j \geqslant 2} 4k^2 \tilde{d}_{\boldsymbol{q}}^{(1)} \tilde{d}_{\boldsymbol{q}'}^{(j)} g_{\boldsymbol{q}'}^{(j)} \partial_{V_{\boldsymbol{q}}^{(1)}} \\
&\times (\langle\!\langle V_{\boldsymbol{q}'}^{(j)} \rangle\!\rangle) Q_s + k^2 \sum_{\boldsymbol{q},\boldsymbol{q}'} \tilde{d}_{\boldsymbol{q}}^{(1)} \tilde{d}_{\boldsymbol{q}'}^{(1)} \partial^2_{V_{\boldsymbol{q}}^{(1)} V_{\boldsymbol{q}'}^{(1)}} Q_s \\
&+ k^2 \sum_{\boldsymbol{q},\boldsymbol{q}',\boldsymbol{q}''} \tilde{d}_{\boldsymbol{q}}^{(1)} \tilde{d}_{\boldsymbol{q}'}^{(1)} \sum_{j \geqslant 2} \left[\frac{\partial_{V_{\boldsymbol{q}}^{(1)}} g_{\boldsymbol{q}''}^{(j)}}{g_{\boldsymbol{q}''}^{(j)}} \partial_{V_{\boldsymbol{q}}^{(1)}} Q_s + \left[\frac{\partial^2_{V_{\boldsymbol{q}}^{(1)} V_{\boldsymbol{q}'}^{(1)}} g_{\boldsymbol{q}''}^{(j)}}{2 g_{\boldsymbol{q}''}^{(j)}} \right.\right. \\
&\left.\left. - \left(\frac{\partial_{V_{\boldsymbol{q}}^{(1)}} g_{\boldsymbol{q}''}^{(j)}}{g_{\boldsymbol{q}''}^{(j)}} \right)^2 \frac{1}{4} - 2 g_{\boldsymbol{q}''}^{(j)} \left[\partial_{V_{\boldsymbol{q}}^{(1)}} \langle\!\langle V_{\boldsymbol{q}''}^{(j)} \rangle\!\rangle \right]^2 \right] Q_s \right].
\end{aligned} \tag{14}$$

We must adjust the Gaussian widths $g_q^{(j)}$ so that relation (14) becomes an equation of continuity for Q_s, i.e. we have a conserved flow of probability on a strip about the CM. The aim is to reduce (14) to a FP equation equivalent to the empirical Langevin equation proposed previously [5]:

$$\frac{\partial \psi}{\partial t} = \frac{1}{\tau_0} [\varepsilon - \tilde{\xi}_0^4 (\nabla + q_0^2)^2 - g\psi^2] \psi + f(\boldsymbol{r}, t) \tag{15}$$

where the inhomogeneous term was regarded in previous treatments of this problem as a *phenomenological* source term. This forcing field *modelled* the effect of the fast hydrodynamic modes which adjust themselves in the adiabatic following.

The parameters in (15) were evaluated (cf [5]):

$$\tau_0^{-1} = \frac{3\pi^2}{2} \frac{\sigma}{\sigma+1} \qquad \xi_0^2 = \frac{8}{3\pi^2} \qquad \xi_0^2 = 4 q_0 \tilde{\xi}_0^4$$

$$\sigma = 0.78 \qquad q_0 d = \frac{\pi}{\sqrt{2}} \qquad g = 0.5.$$

The radius of the cell under consideration is $L = 4.72$. The order parameter is defined as follows:

$$\psi = c \sum_{q} V_{q}^{(1)} \exp(\mathrm{i}\boldsymbol{q} \cdot \boldsymbol{r}).$$

The constant c is obtained from the following relations (cf [5]):

$$c_q^{(1)}(r, z) = \frac{1}{\bar{c}} \begin{pmatrix} \mathrm{i}q\tilde{u}_c(z) \\ \alpha_0(z) \\ \theta_0(z) \end{pmatrix} \exp(\mathrm{i}qx) \tag{16}$$

where

$$\alpha_0(z) = 4\mathrm{i} \cos \pi z \qquad \alpha_0(z) = 2\sqrt{2} \sin \pi z \qquad \theta_0(z) = 9\sqrt{2}\pi^3 \sin \pi z. \tag{17}$$

Thus

$$\begin{aligned} \bar{c} &= [\sigma(\theta_0^2)_m + R_c(|u_0|^2 + w_0^2)_m]^{1/2} \\ c &= [(\omega_0\theta_0)_m / R_c]^{1/2}\bar{c}^{-1}. \end{aligned} \tag{18}$$

Thus the CM-reduced equation which corresponds to the Langevin equation (15) is

$$\partial_t Q_s = -\sum_{q} \partial_{V_q^{(1)}}\{\langle\langle \dot{V}_q^{(1)} \rangle\rangle Q_s\} + k^2 \sum_{q,q'} \tilde{d}_q^{(1)} \tilde{d}_{q'}^{(1)} \partial^2_{V_q^{(1)} V_{q'}^{(1)}} Q_s. \tag{19}$$

In order to obtain (19) from (14), we have to introduce the following scaling relations:

$$g_q^{(j)} = \frac{\tilde{g}_q^{(j)}}{(\varepsilon/\tau_0)^2} \qquad k = G^{-1/2}\tilde{k}. \tag{20}$$

The quantities $\tilde{k}$ and $\tilde{g}_q^{(j)}$, j bigger than 1, are of O(1). We also have

$$G = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} g_{q_0}^{(j)}. \tag{21}$$

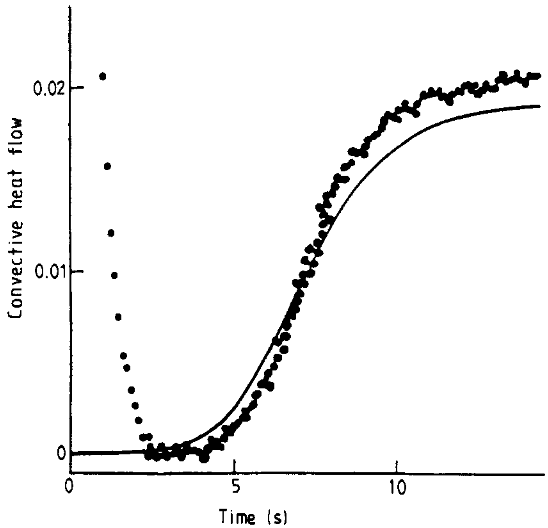

Figure 1. Integration of the CM-reduced Fokker-Planck equation (19). Use is made of the scaling relations (20)-(22) and the normalised parameters are taken equal to 1. The full curve gives the theoretical predictions for the convective heat flow as defined by equation (8). The experimental data were obtained from [5].

Then, to $O(\varepsilon)$, (14) reduces to (19) if and only if

$$g_q^{(j)} = -\lambda_q^{(j)}/(k\tilde{d}_q^{(j)})^2 \tag{22}$$

where $\lambda_q^{(j)-1}$ is the characteristic relaxation time for the mode $V_q^{(j)}$.

We proceed now to integrate (19) making use of the scaling equations (20)-(22) and choosing the normalised parameters equal to 1. The full curve in figure 1 corresponds to the convective heat flow as given by (8). The experimental data can be found in [5]. The size of the step in the heat input determines the unfolding small parameter: $\varepsilon = 0.049(1+\pi^2\xi_0^2/L^2)+\pi^2\xi_0^2/L^2$.

The intensity of the random source term $f(r, t)$ was adjusted in previous approaches [5] in order to model the effect of fast hydrodynamic modes on the transition to a convective state. It has been shown [5] to differ considerably from the value obtained for equilibrium thermal fluctuations. According to this work (see also [4]), the strength of the random source should not be regarded as an adjustable parameter; rather, the onset of a CM implies that this parameter is dependent on the Gaussian width of the probability distribution about the CM as shown by (20)-(22).

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